Only you can tell me: Is this *Handouts* "clear, brief, easily assimilated by all"?

Quadratic Equations

'SQUARE' — a SHAPE and a PROCESS

A square is the **simplest shaped quadrilateral,** a quadrilateral with four equal sides (*quad*=four; *latus*=a side) and four right angles, forming a quadrangle.

Its area is the **square of the number** representing the length of a side. Squaring a number is multiplying it by itself and written in mathematics as x^2. They say it's raising x to the power of two.

That is why quadratic equations are named from 4, though the highest power of x in them is 2.

A quadratic equation is like this.

$5x^2 + 3x - 1 = 0$

The numbers 5 and the 3 are coefficients; -1 is the constant and not a modifier of x^2 or x.

The general form of a quadratic equation is:

$$\mathbf{ax^2 + bx + c = 0}$$

where x is a variable and a, b and c are fixed numbers for that particular problem.

REVISION — PERFECT SQUARES

Binomials means two algebraic numbers added or subtracted from each other as in these perfect squares:

$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$

But (a-b) multiplied by (a+b) is $(a-b)(a+b) = a^2-b^2$. This is not a perfect square, rather the difference (subtraction) of perfect squares. The sum (addition) of perfect squares is $a^2 + b^2$ and not a perfect square either and has no real factors.

Without familiarity with these binomial facts, you'll limp and stumble forever! Prove them! Spot their 'shapes' with a & b; or b & a; h & k; x & y.

Also know and understand and be able to prove: $(a+b)(c+d) = ab + ac + bc + bd$; $(x+a)(x+b) = x^2 + (a+b)x + ab$ and all their variants with + and - mixed. They must not confuse you.

CHECK $(a - b)^2$ with numbers like x=4, y=3. If a particular case won't work, the algebra is wrong:

LHS: $(4-3)^2 = 4^2 - 2\times4\times3 + 3^2 = 16 - 24 + 9 = 1$ and also $(4-3)(4-3) = 1 \times 1 = 1$ ——— Q.E.D.

Check (a-b)(a+b): LHS: $(4-3)(4+3) = 1 \times 7 = 7$ and RHS: $4^2-3^2 = 16 - 9 = 7$ ——— Q.E.D.

Q.E.D. *Quod erat demonstrandum* "What was to be proved" (or for those who limp without Latin: "Quite Easily Done").

Perfect squares in quadratic equations

The easiest to solve quadratic equation is a perfect square of a binomial like $(a+b)^2$. You must be able to factorize such things on sight and instantly: e.g.

$x^2 - 6x + 9 = 0$ ——————— (1)

by recognizing the binomial factors give $(x-3)^2 = 0$.

$\therefore x = 3$.

Check: substitute x=3 in (1):

LHS: $3^2 - 6\times3 + 9 = 9-18+9 = 0$ = RHS — Q.E.D..

FACTORIZING QUADRATICS

Example: $x^2 - 4x + 3 = 0$ ——— (1)

$\therefore (x-3)(x-1) = 0$

$\therefore$ Either x = 3 or else x = 1.

Check by substitution of these answers in (1):

LHS: check x=3: $3^2 - 4\times3 + 3 = 9-12+3 = 0$ = RHS also check x=1: $1^2 - 4\times1 + 3 = 1-4+3 = 0$ = RHS.

Note the way we found the factors: the factors of +3 are $(+3\times+1)$ and (-3×-1). Also, the factors have to add up to -4: hence the choice of -3 and -1.

WHAT IF IT WON'T FACTORIZE?

Example: $x^2 + 4x - 3 = 0$ ——— (1)

The factors of -3 are $(-3\times+1)$ and $(+3\times-1)$: they multiply to give -3 but they don't add to give +4.

What do we do? We use cunning…

COMPLETING THE SQUARE

To a new equation $x^2 + 4x - 3 = 0$, —— (1) add and subtract 4. This won't change it, but lets a perfect square to be made of $x^2 + 4x$; this 4 is squaring half the coefficient of x, actually $(4/2)^2$.

$x^2 + 4x + 4 - 4 - 3 = 0$

re-grouping,

$[x^2 + 4x + 4] - 4 - 3 = 0$;

get ready to square root: factorize, group, transpose:

$(x + 2)^2 = 7$

$\therefore x + 2 = \pm\sqrt{7}$

$\therefore x = -2 \pm \sqrt{7}$

Check answer: first, substitute $x=(-2+\sqrt{7})$ in (1):

LHS: $(-2+\sqrt{7})^2 + 4(-2+\sqrt{7}) - 3$

$= (4 - 4\sqrt{7} + 7) - 8 + 4\sqrt{7} - 3$

$= 4 + 7 - 8 - 3 = 11 - 11 = 0$

RHS = 0 ——— Q.E.D.

Next, substitute the other answer $x=(-2-\sqrt{7})$ in (1):

LHS: $(-2-\sqrt{7})^2 + 4(-2-\sqrt{7}) - 3$

$= (4 + 4\sqrt{7} + 7) - 8 - 4\sqrt{7} - 3 = 11 - 11 = 0$

RHS = 0 ——— Q.E.D.

TEACHERS

There is far more than one lesson on these two pages. With necessary revision and working out lots of problems, there are lesson ideas here for weeks.

Pupils need to memorize, understand and work out lots of examples. Understanding needs memory and memory needs understanding. The two work together. They're God-given powers of the human mind in constant mutual interaction in a symbiotic relationship. What God has given, let not man discard.

Understanding requires proving formulas to help memory function. See *Handouts* n. 126.

That is why the solution of quadratic equations is proved rigorously on the next page.

Solving the General Quadratic Equation

$$ax^2 + bx + c = 0 \quad \text{————————} \quad (1)$$

in which a and b and c are numbers **which constrain the possible values** for x, which we have to discover.

FIRST, make coefficient of x^2 in (1) unity, thus:
(1) $\div$ a : $\quad x^2 + (b/a)x + c/a = 0$

COMPLETE THE SQUARE

on the two terms which are powers of x:

$$x^2 + (b/a)x + (b/2a)^2 - (b/2a)^2 + c/a = 0 \quad \text{——} \quad (2)$$

Explanation: we have added and subtracted the same amount $(b/2a)^2$. The 2 in the 2a is needed to offset the 2ab as in the expansion of $(a+b)^2$.

RE-GROUPING

Re-group the terms in equation (2):

$$[x^2 + (b/a)x + (b/2a)^2] - [(b/2a)^2 - c/a] = 0 \quad .. \quad (3).$$

The minus for c/a offsets the minus outside the brackets.

$$\therefore\ [x+b/2a]^2 - [(b/2a)^2 - c/a] \ = \ 0 \quad \text{——} \quad (4)$$

Transposing:

$$[x+b/2a]^2 = [(b/2a)^2 - c/a]$$
$$= b^2 / 4a^2 - c/a$$
$$= \frac{b^2 - 4ac}{4a^2} \quad \text{————————} \quad (5)$$

Take square roots of both sides in (5):

$$[x+b/2a] = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Transpose for the solution:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b\pm\sqrt{b^2 - 4ac}}{2a} \quad \text{—————} \quad \text{Q.E.D.}$$

MATHS IS GOOD FOR YOU !

TEACHERS, though circumscribed by the capabilities of the students, can make maths...

- **interesting;**
- **relevant;**
- **memorable,** and tested for practical understanding;
- **understandable,** by patiently proving it;
- with **special help** ***for those verging on despair!***
- Memory is the first essentail for practical usefulness.
- Understanding helps accurate remembering.
- Understanding is a big help in practical use.
- Stack numbers in neat columns for adding up.
- Always add column of figures **twice:** down, then up.
- Always check subtraction by addition.

LINKING apparently unrelated studies, such as mathematics with literature. For instance:

Imaginary Numbers are not as silly as they sound.

They combine the idea of a number's square root with the idea of a negative number.

Each idea on its own matches up with real life: Finding length of a side for a square paddock of a set area. Measure depth under water by a negative.

This prompts the idea of a square root of a negative. Note that there are two roots of $\sqrt{+1}$: either plus one or minus one (written as $\pm$ 1).

But can minus numbers have square roots? While $\sqrt{9} = 3$ (or even - 3), what does $\sqrt{-9}$ mean? i.e. what multiplied by itself equals - 9 ? The answer is not 3 but 3i where $i = \sqrt{-1}$ which is **imaginary.**

So $\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = i^2 = -1. \quad \therefore\ i^3 = -i$!

Does this *seem* nonsense? It does not match with anything our five senses perceive in the real world. Rather, it's **like a fantasy story** for which we willingly imagine splendours in the distant past, or weird doings in the distant future in science fiction. It's more desirability than possibility, but fantasy can help virtue.

So too in maths: imaginary numbers are powerful.

Imaginary numbers stretch our imagination, extend our understanding of mental possibility. And they are powerful and practical in their applications in higher mathematics.

A little mental exercise with a pencil: the cube roots of -1: obviously, $-1\times-1\times-1 = -1$. Now try cubing $(-1 + \sqrt{3}i)/2$ and then $(-1 - \sqrt{3}i)/2$.

Coming down to earth...

MANY pupils, primary and secondary, need a concrete overture to mathematics, less abstract, with lots of seeing, handling and measuring as in science experiments. This makes mental digestion easier than do abstract diagrams drawn on paper or white-boards.

A teacher can do essential theory afterwards: then it's built on familiarity and much more palatable.

Hands-on home-made cut out models in wood or cardboard help pupils with geometry to get a grasp on the closely reasoned theorems of Euclid.

These theorems are most useful: e.g. triangulating the frames of buildings so they don't "slurp" like the Leaning Tower of Pisa, or fall over !