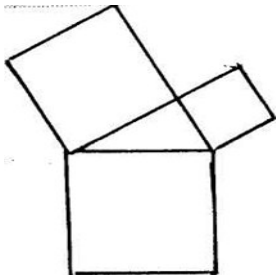
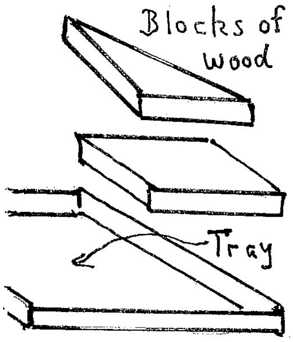
Pythagoras' Theorem



In a right-angled triangle the square on the hypotenuse equals the sum of the squares on the other two sides. i.e. the area of the square on the long side (opposite the right angle) equals the sum of the areas of the squares on the other two sides.

1. START WITH A JIGSAW PUZZLE

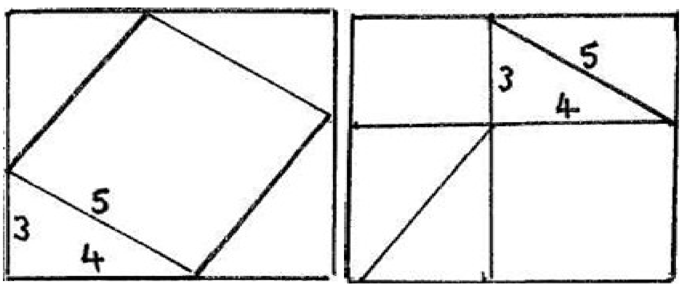
Discover Pythagoras' Theorem over ten years (5 to 15) with a jigsaw puzzle.



In 19mm pine, make:

- **eight triangles** (8Δs) each with handy-size sides 3, 4, 5 inches (or centimetres, or 6,8,10 cm). Mark the sides of two of the Δs with 3,4,5. Pupils should be led to find that the secret of such Δs is the 90° ∠ (angle) opposite the longest side; (don't worry them with 'hypotenuse').
- **three squares**, one for each side of the 3,4,5 Δ;
- **two shallow trays,** each a 7 inch square.

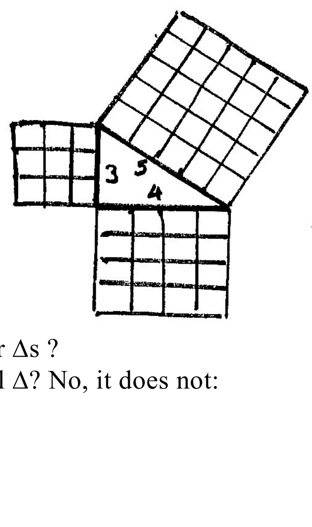
Pupils try to fit all 8 Δs and 3 squares into the two trays (and so discover some "rules of packing"):



- Put the biggest in first: discover that the 5 inch square will not fit in the same tray as the 3 and 4 inch squares.
- With shapes like Δs (and other odd shapes) two sorts of rotations help in packing: *turning them over* (upside down) and *turning them round* through various angles.

2. THE PARTICULAR CASE OF 3, 4 & 5

- The two trays are equal size (each 7x7) so each's total area of squares and triangles is the same.
- Take 4 Δs from each tray: ∴ what is left is equal: ∴the 5 inch square = 3 inch square + 4 inch square.
- Check this: mount the squares on the sides of a 3,4,5 Δ. Mark small boxes 1x1 on all three squares. Calculate $3^2+4^2=5^2$; and/or count them, 9+16 = 25.



- Check again: weigh the Δs on kitchen scale. The weight of the small Δs adds up to the big one.
- Is this 3 + 4= 5 just a lucky chance and not true for any other Δs ?
- Does it work for any equilateral Δ? No, it does not: $1^2 + 1^2$ does not equal 1^2
- Does it work for any 90° Δ?

Further investigation is needed

3. SCIENTIFIC METHOD - EXPERIMENT

Draw several 90°Δs and carefully measure the sides a,b,c, where side c is called the **hypotenuse,** a word meaning "stretched under", i.e. opposite the 90°.
Tabulate the results thus and do the calculations:

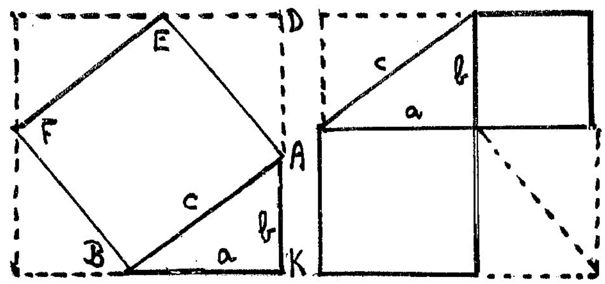
90°Δ	a	b	c	a^2	b^2	$a^2 + b^2$	compare c^2
(1)							
(2)							
(3)							
(4)							

By this experiment (a case of scientific method), it would seem that as "a working proposal" *(hypothesis)* that, in any 90°A, $a^2 + b^2 = c^2$, i.e. **in any right-angled triangle, the square on the hypotenuse equals the sum of the squares on the other two sides.**

4. STRICT PROOF of Pythagoras' Theorem

Nos. 2 and 3 are not *mathematical* proofs: no. 2 is a *demonstration;* no. 3 is an experimental proof as used in science, but it lacks the rigour of mathematics.

Proof for any 90°Δ: Flip over the two Δs marked 3,4,5. Mark them a,b,c to *represent* any 90°Δ, no longer a



particular triangle, but "the general case":

(1) On Left Hand Side (LHS) there are four congruent Δs each a, b, c packed round "c-square". (K is used instead of capital C to avoid confusion with the side little c). First, we must prove that this combined figure is really a square.

By construction, Δs KBA and DAE are congruent; .-. ∠ KBA = ∠.DAE

Add up the 3 angles at A: ∠DAE + ∠EAB + ∠KAB = ∠KBA + 90° + ∠KAB ;
= 180° (because these are the 3∠s of the ΔKBA)
. . at A, KAD is a straight line; similarly at B, F & E.

(2) ∴ The LHS is a big square with sides a+b, and area $(a+b)^2$;.
now the RHS is also a big square because of all the 90° ∠s and it also has sides a+b, and area $(a+b)^2$;
∴ **area on LHS = $(a+b)^2$ = area on RHS.**

(3) Area of LHS = "c-square" + 4Δs
= $c^2 + 4(ab/2) = \mathbf{c^2 + 2ab}$ (since the area of a triangle is half of its base times its height = ab/2).

Also area on RHS = "a-square" + "b-square" + 4 Δs
= $a^2 + b^2 + 4(ab/2) = \mathbf{a^2 + b^2 + 2ab}$;

Next, subtract 2ab from both LHS and RHS:

∴ $a^2 + b^2 = c^2$ Q.E.D.

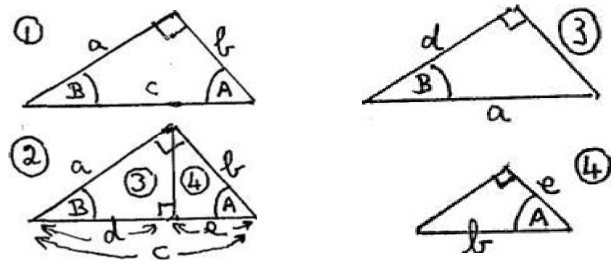
(Q.E.D. = *Quod Erat Demonstrandum* = "what was to be proved" — or in dog-Latin, "Quite Easily Done").

Corollary: Note that the two expressions for the area of

5. ANOTHER STRICT PROOF

Similar triangles also provide a rigorous proof.

Note that if two Δs have three matching sides equal, they are **congruent**, so their matching angles are equal. **The converse is not true**: if two triangles are **similar** (i.e. with matching angles equal) then their **sides will be in the same proportion** (or ratio), but are not necessarily equal.



Make the abstract more concrete using **cardboard cut-outs** of two identical 90° Δ*s* (1) & (2) and mark ∠s A,B and sides a,b,c. Draw a perpendicular to cut Δ(2) into two 90°Δs (3) & (4). Note that d+e=c. "Flip over" Δs (3) & (4) and rotate them a little so that their 90° ∠*s* are at the top, and their hypotenuses at the bottom. Note that Δs (1), (3) & (4) are ***similar***. (In fact, check this by stacking them on top of each other with the 90° touching). By using pairs of ∠*s* A,B,90° in pairs of Δs, pick out the ratios of the sides:

In Δs (1) & (3): a/c = d/a. $a^2 = dc$

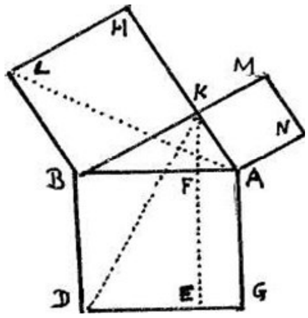
In Δs (1) & (4): b/c = e/b. $b^2 = ec$

∴ $a^2+b^2 = dc + ec = (d+e)c$ But d+e = c

∴ $\mathbf{a^2 + b^2 = c^2}$ **Q.E.D**

6. EUCLID'S PROOF

This best-known of the proofs is harder than nos. 4 or 5. Join LA, & KD; draw KE at 90° to AB & GD. Δs LBA & KBD are congruent (2 sides, incl.∠.); Δ LBA is half the square LBKH (on same base and between same parallels, and similarly Δ KBD is half the rectangle BDEF.



∴ square LBKH=rect. BDEF. By a similar construction, square KANM = rect. AGEF.

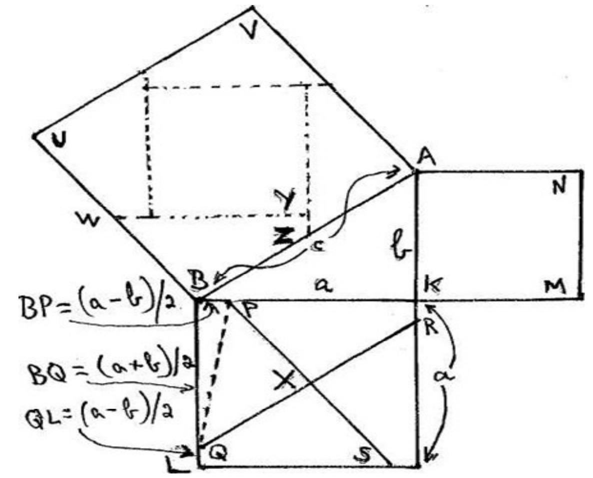
∴ squares LBKH + KANM = square ABDG Q.E.D.

7. A HARDER JIGSAW PUZZLE

Here is another very convincing *demonstration* of Pythagoras' Theorem. Draw on thick cardboard (or saw up wooden blocks) a 3,4,5 Δ (ideal dimensions to get three very different sized squares). Construct the squares on all three sides and divide "a-square" into 4 quadrilaterals, as in the picture, with a= 4, b=3, so that the crucial distance BP = (a-b)/2 = 1/2. Fit them as jigsaw pieces into the corners of the big square. (Hint: do not rotate the quads but move them so that tthe 90° ∠ at X slide into die comers of c-square). Cut out b-square and fit it in the middle. Therefore the big square on the hypotenuse is equal to the sum of the squares on the other two sides. Q.E.D.

Outline of a strict proof for no. 7

In the 'a-square' plot points P,Q,S,R so BP,QL,SH, and RK = (a-b)/2. Join P to S and Q to R. (1) **BQRA is a parallelogram** because BQ=AR [BQ = a-(a-b)/2 = (a+b)/2, and AR=b+(a-b)/2 = (a+b)/2] and BQ//AR [sides in a-square; // means "is parallel to"].



∴ hypotenuse side c of AABK = AB = QR

∴ c/2 = XQ = XR = XP = XS

(2) $\mathbf{a^2}$ **= area "a-square"** = area of 4 quadrilaterals
= area of 4 Δs like BQP + area of 4 Δs like XQP
= 4[(l/2)(a+b)/2 x (a-b)/2] + 4[(l/2)(c/2)(c/2)
= $(a^2-b^2)/2 + c^2/2$ .... and transpose terms to get that:
$\mathbf{a^2 + b^2 = c^2}$ Q.E.D.

(3) But do their shapes fit as a jigsaw into c-square? Move quad PXRK to WBZY, and quad XQLS to V, and quad XRHS to U, and quad XPBQ to A. The opposite angles of these quads are supplementary, so an adjacent pair make a straight line. ∴ The 4 quads really fit.

Alternative construction of the 4 quads: join the diagonals in 'a-square' to find its centre at X. Draw a perpendicular and a parallel to the hypotenuse-c to get PS & QR.

8. CONVERSE of PYTHAGORAS THEOREM

The converse of Pythagoras' theorem is also true: **A triangle with sides a, b, c such that $a^2 + b^2 = c^2$ has a right-angle opposite the side 'c'.**

Proof: If it were not 90°, then we could draw another Δ which does have 90° and turns out to be congruent with the a,b,c A.

9. INTERESTING TRIANGLES

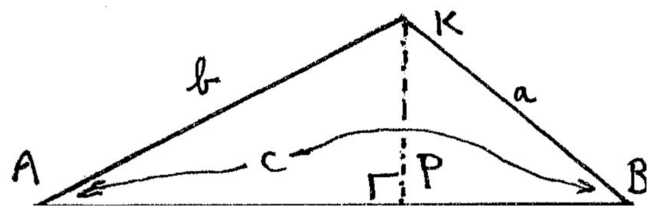
More Pythagorean Δs with whole numbers:

Δs	P	Y	T	H	A	G	O	R	E	A	N
a =	3	5	7	8	9	11	12	13	15	16	17
b=	4	12	24	15	40	60	35	84	112	63	144
c =	5	13	25	17	41	61	37	85	113	65	145

N.B. After 3,4,5 and 5,12,13, the Δs are too skinny for any practical accuracy in constructing exact right-angles in woodwork etc.

10. NON RIGHT-ANGLED TRIANGLES

Any triangle can be divided into two right-angled triangles by drawing a perpendicular from the vertex to the base:



We can use Pythagoras' Theorem in As APK, BPK to get the **Cosine Rule**: a2 = b2 + c2 - 2bc(cosA) where cosine A or cos A = AP/AK (which is trigonometry).

TO BE CONTINUED in *Handouts* no. 19 on Trigonometry.