Trigonometry

TRIGONOMETRY, Greek metros+trigon> means "the measure of a triangle", the measure of three angles; cf. pentagon (five angles); hexagon (six angles).

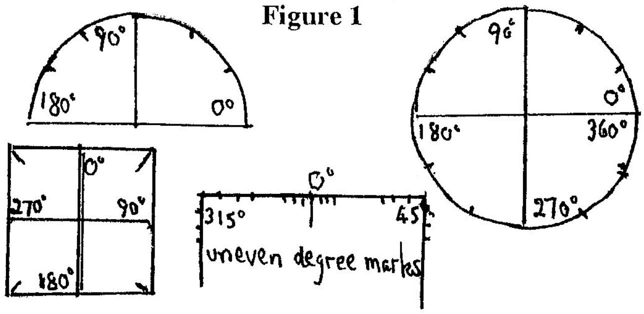
Angles are measured in degrees (°) — a small circle ° is apt for how many $\frac{1}{360}$ parts of a full circle of 360°.

The Babylonians liked 360 since it factorizes neatly:-
360 = 6 x 60; and 6 = 2 x 3 and 60 = 3 X 4 x 5.

PROTRACTORS

The easiest way to measure angles drawn on paper is a protractor. Protractors for students are usually semi-circular, from 0° to 180°, though news agents also stock full-circle protractors. Navigators often use square protractors with the degree marks more bunched in the middle of sides, and more spread out at corners:-

Figure 1



Protractors are useful on plans and maps which are drawn to scale. Roughly drawn pencil sketches and mud maps are not to scale and need some way of *calculating* angles. Angles measured in degrees cannot be used for direct calculating.

ANGLES ∠s AND TRIANGLES Δs

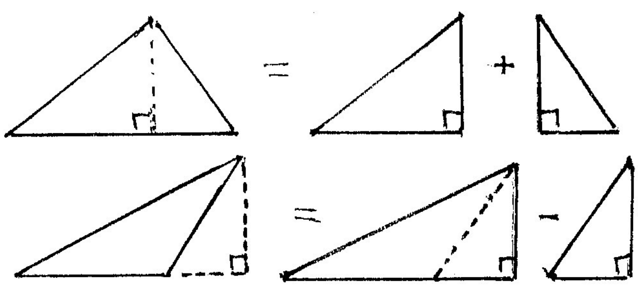
TRIGONOMETRY is a powerful tool for working with angles. It ***calculates*** angles that protractors can't measure accurately enough, or not measure at all. Triangles are the key to a mastery of angles. Any angle less than 180° can be drawn in a triangle. Any angle greater than 180° can be matched up to an acute (= sharp) angle less than 90°.

A triangle with a 90° angle is called a right-angled triangle, or a "right triangle", and abbreviated 90°Δ.

Any triangle can be cut into two 90°Δs simply by drawing a perpendicular (drawing, not dropping!) from a vertex to the opposite side. These two 90°Δs add up to the original triangle. For a triangle with an obtuse (= blunt) angle, one subtracted from the other makes the original triangle. Also the two acute angles in a 90° Δ are complementary, that is, they *complete* each other by adding up to 90°.

THE "SINE" OF AN ANGLE

Figure 2



An angle Φ (say phi or fi, as in 'Fee, Fi, Fo, Fum') is made up of two arms, i.e. two intersecting lines.

Mark points R and T on these lines equidistant from the vertex O. Draw the arc of a circle with radius OR. This radius will be our unit measure of length. Thus OR=OT = 1

The curved arc is itself a measure of the angle Φ — but a straight edged ruler cannot directly measure an arc. So draw a perpendicular from R to OT to meet it at P, as in Fig. 3.

The length of PR measures Φ indirectly. And as Φ increases so does PR, though not at the same rate.

Draw Figure 3 to scale in a work book, and label it.

Figure 3

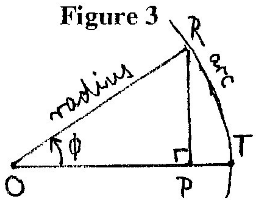
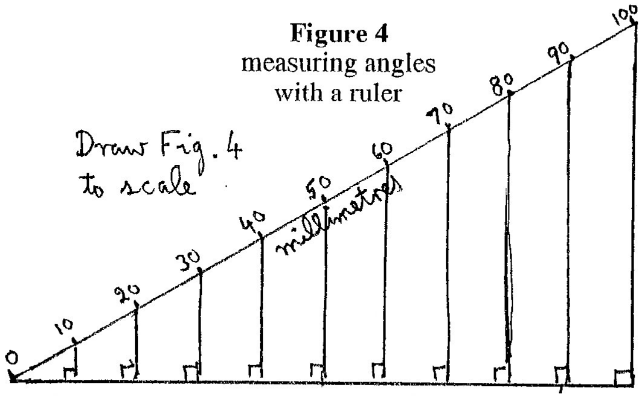


Figure 4
measuring angles
with a ruler



For easy measurements, start with a 30° angle at the left end of a base line. On the upper line mark spaces at 10mm up to 10cm. Draw a perpendicular from each mark to the base.

Measure the perpendiculars in mm and tabulate against the ten long sides. Long sides in 90°Δs are named **hypotenuse**(Greek: 'stretching below' or opposite the ∠90°). The perpendiculars are the **opposite** side to the 30°. Divide each opposite by its hypotenuse. Are the ratios always 0.5 ?

opp.	5									50
hyp.	10	20	30	40	50	60	70	80	90	100
ratio	0-5									0-50

This suggests that 0.50 is a constant for all 30° angles.

Draw a new figure with 60°. Is the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ 0.87 ?

Is there a special ratio for every angle?

Doubters can make sets of Δs and measure other sized angles.

We define the **sine of an angle Φ** in a 90°Δ as the ratio of **"opposite over hypotenuse"** and write it shorthand as sin Φ.

Thus we can measure angles by sines as well as by degrees.

LEARN BY ROTE: In the 90°ΔABH, with sides a, b, h, and ∠A = Φ, ∠H = 90°, **sine** Φ = $\frac{\text{opposite}}{\text{hypotenuse}}$ = **a/h**. (Avoid using c: written, it looks like C).

Hold Fig. 5 in your imagination.

Do lots of problems with sine from a book. Use calculator and/or tables.

Figure 5

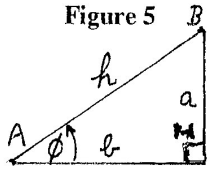
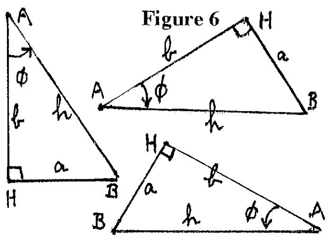


Figure 6

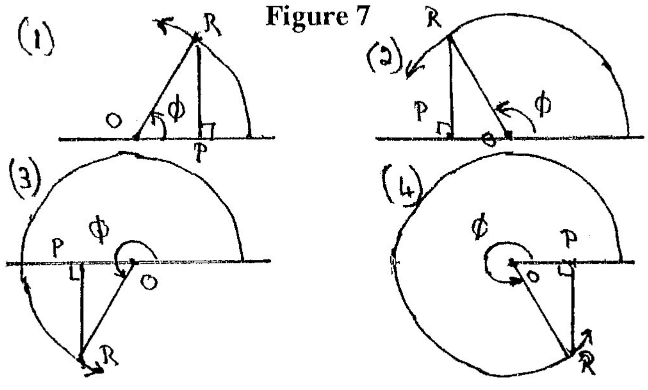


In Figure 6, three 90°Δs, each with the ∠ Φ, are rotated or mirrored. Don't limp with acronyms! Name sine Φ merely on seeing a 90° Δ.

A NEW DEFINITION for sine if Φ > 90° or > 180°.

In Fig. 7 we extend Fig. 3 by continuing the arc of unit radius anti-clockwise, to bring in the other three quadrants:-

Figure 7



In all quadrants, the sine of an angle between two radii of a circle is the ratio of a vertical over a radius, where the vertical stands on one radius and reaches to the end of the other.

This new definition **extends , not contradicts** the first one.

Since sin Φ = PR/OR, and when we define radius OR = 1, for each angle Φ above, the vertical PR can *represent* sin Φ.

The x-axis on the right of O is positive, negative to the left; and the y-axis is positive above the x-axis, negative below.
By definition, the radial arm OR is always positive.

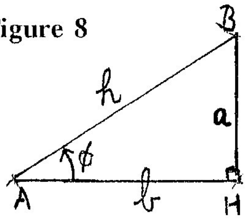
In Figures 7(3) and 7(4), PR is negative, so sin Φ is negative in both the 3rd and 4th quadrants.

Latin sinus means arc (or curve or breast); sin Φ measures how much of the circumference the arc curves around, and so it indirectly measures the angle subtended at the center by that arc.

COSINE

Figure 8 uses a, b, h from Fig. 5. We define the **cosine** of Φ as the ratio $\frac{adjacent}{hypotenuse}$ and write it **cos Φ = b/h**. "Cosine" is apt: the side adjacent Φ is opposite 90° - Φ, the complement of Φ, and so a cosine "complements the sine" (Latin co– together).

Figure 8



In Fig. 7, cos Φ = OP/OR, OR = 1, so adjacent side OP represents cos Φ.

Hence the cosine is negative in the 2nd and 3rd quadrants because OP is negative. It is positive in 1st and 4th quadrants.

Now work out lots of cosine problems from a book.

TANGENT

In the Figure 10 (next column), draw at T a tangent to the arc. Let it meet OR produced at S. ΔOTS is a 90°Δ.

In 90° ΔOTS, we define the **tangent** of Φ as the ratio of TS, opposite Φ, to adjacent side OT: **tan Φ = a/b** (of Fig. 8) = $\frac{opposite}{adjacent}$
Since radius OT= 1, tan Φ can be *represented* by TS.

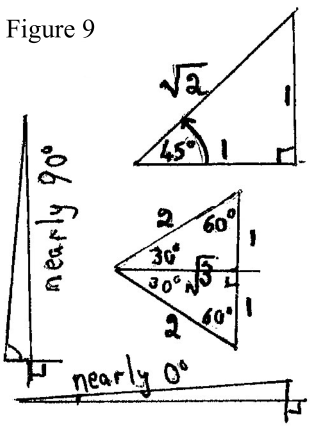
The name tan Φ is apt because a tangent is a line which touches a circle only once (Latin, tangere "to touch").

In Fig. 7, the tangent is negative in the 2nd quadrant because OP is negative; and also negative in the 4th quadrant because PR is negative. In the 3rd quadrant both PR and OP are negative so PR/OP is positive. In the 1st quadrant all ratios are positive. Now work lots of tangent problems from a book. From Fig. 8 we prove tan Φ = $\frac{\sin\Phi}{\cos\Phi}$

MEMORIZE table by picturing Figure 9 and prove by Pythagoras:-

∠	0°	30°	45°	60°	90°
sin	0	1/2	1/√2	√3/2	1
cos	1	√3/2	1/√2	1/2	0
tan	0	1/√3	1	√3	∞

Figure 9



RECIPROCALS

Reciprocals of sine, cosine, tangent in Fig. 8 are defined as cosecant, secant, cotangent, and written as cosec Φ=h/a, sec Φ=h/b, and cot Φ=b/a.

PYTHAGORAS in TRIG: Prove from Fig. 8 that:-
$\sin^2\Phi + \cos^2\Phi = 1$, then, by dividing by $\cos^2\Phi$ and $\sin^2\Phi$, that $1 + \tan^2\Phi = \sec^2\Phi$ and $1 + \cot^2\Phi = \text{cosec}^2\Phi$.

SECANT = sec Φ

In Fig. 10, sec Φ = OS/OT; OT = 1, OS *represents* sec Φ.

Lines like OS cutting a circle are called secants (Latin, secure, to cut), hence the name secant for this ratio.

COSECANT = cosec Φ

In Fig. 11, tangent ST is produced to U so that ∠SOU = 90°.

In 90°ΔOTU, ∠OUT is the complement of ∠TOU, itself the complement of ∠TOS, i.e. of Φ, so ∠OUT = Φ

Thus in Fig. 11, cosec Φ = OU/OT. However, OT = 1, so OU *represents* cosec Φ.

Figure 10
(in four steps, make two Δs above OT)

1. At a point O, make ∠ Φ.
2. Mark P, R on arms of Φ so that PR is 90° to OP to make 90°Δ OPR, the base Δ.
3. Make an arc with radius OR to meet OP produced at T
4. Tangent at T to this arc meets OR produced at S, thus ΔOTS is a 90°Δ.

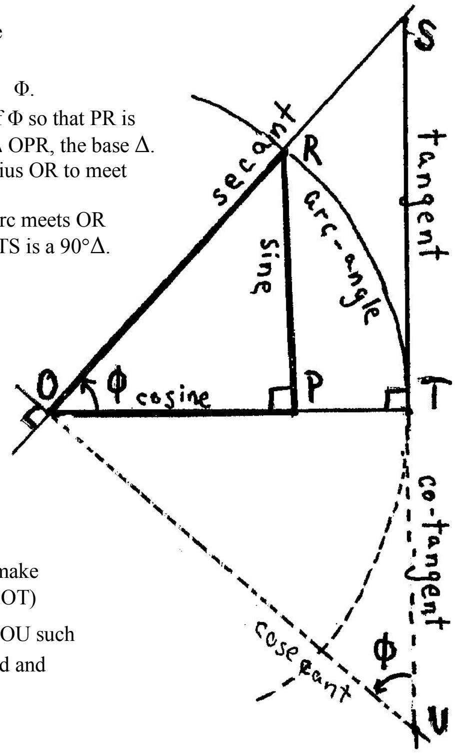


Figure 11
(in one more step, make the dotted Δ below OT)

5. At O make a 90° ∠ ROU such that U is on ST produced and

Though sine is more basic than cosine, secant is more basic than cosecant hence the switch-over of the prefix 'co'.

COTANGENT = cot Φ

In Figure 11, in the 90°ΔOTU, cot Φ = TU/OT. But OT= 1. Therefore TU *represents* cot Φ.

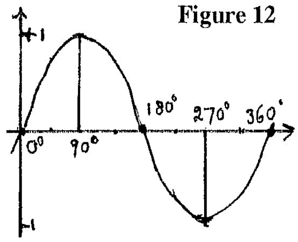
Contrast in trig ratios of complement of Φ

Though sin Φ = cos (90°- Φ) and sec Φ = cosec (90°- Φ) cot Φ = tan (90°- Φ). Thus the cotangent of an angle is the tangent of its complement, hence the name cotangent.

FURTHER TRIG

- Graph x = Φ, y = sin Φ. Rules for + and - in the four quadrants of Fig. 7 produce a wave-form. Also draw the graphs of cosine and tangent. Research **Lissajous Figures** where sines of various phases & frequencies are graphed as x & y; e.g. ABC news logo.

Figure 12



- Teach yourself **spherical trig**, in which sides of spherical Δs are measured in degrees, being arcs of Great Circles on a sphere.
- **Haversine** (half a versine): hav <t> = (1- cos Φ)/2. It runs from 0 to 1 as Φ runs from 0° to 180°, where, unlike cosine, it is never negative, and unlike sine, never repeats itself, so it is easier for navigators lacking a calculator to solve the spherical Δs. In Figure 11, from T to half-way to P represents hav Φ.

Please ring Father Tierney 02 4829 0297 if you need help.